# Curvature inequalities for Operators in the Cowen-Douglas class of a Planar Domain

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## Abstract

Fix a bounded planar domain $\Omega$. If an operator $T$, in the Cowen-Douglas class $B_1(\Omega)$, admits the compact set $\overline{\Omega}$ as a spectral set, then the curvature inequality $\mathcal{K}_T(w) \leq -4\pi^2 S_\Omega(w,w)^2$, where $S_\Omega$ is the Sz̈ego kernel of the domain $\Omega$, is evident. In particular, for any contraction $T$ in $B_1(\mathbb{D})$, $\mathcal{K}_T(w) \leq -4\pi^2 S_{\mathbb{D}}(w,w)^2 = -(1-|w|^2)^{-2}$. The curvature of $U_+^*$, the standard unilateral backward shift operator, is $\mathcal{K}_{U_+^*}(w) = -(1-|w|^2)^{-2}$, $w \in \mathbb{D}$. However, it is easy to construct examples of contractive operators $T$ in $B_1(\mathbb{D})$ for which $\mathcal{K}_T(w_0) = -(1-|w_0|^2)^{-2}$ for some $w_0 \in \mathbb{D}$ but $T$ is not unitarily equivalent to $U_+^*$.

After imposing some "mild" conditions on the class of co-subnormal contractions $T$ in $B_1(\mathbb{D})$, it is shown that if $\mathcal{K}_T(w_0) = -(1-|w_0|^2)^{-2}$ for an arbitrary but fixed point $w_0 \in \mathbb{D}$, then $T$ is unitarily equivalent to $U_+^*$.

Except when $\Omega$ is simply connected, the existence of an operator for which the equality namely, $\mathcal{K}_T(w) = -4\pi^2 S_\Omega(w,w)^2$ for all $w$ in $\Omega$, holds is not known. However, one knows that if $w$ is a fixed but arbitrary point in $\Omega$, then there exists a bundle shift of rank 1, say $S$, depending on this $w$, such that $\mathcal{K}_{S^*}(w) = -4\pi^2 S_\Omega(w,w)^2$. It is proved that these *extremal* operators are uniquely determined: If $T_1$ and $T_2$ are two operators in $B_1(\Omega)$ each of which is the adjoint of a rank 1 bundle shift and $\mathcal{K}_{T_1}(w) = -4\pi^2 S_\Omega(w,w)^2 = \mathcal{K}_{T_2}(w)$ for some fixed $w$ in $\Omega$, then $T_1$ and $T_2$ are unitarily equivalent. A surprising consequence is that the adjoint of only some of the bundle shifts of rank 1 occur as extremal operators in domains of connectivity $\geq 1$. These are then described explicitly.

For a tuple of commuting operator $\boldsymbol{T} = (T_1, \ldots, T_m)$ in $B_n(\Omega)$, where $\Omega$ is a bounded domain in $\mathbb{C}^m$, a curvature inequality is found.

The module tensor product of a Hilbert module $\mathcal{H}$, induced by a Cowen-Douglas class operator $\boldsymbol{T}$, and a two dimensional Hilbert module over the function algebra $\mathcal{O}(\overline{\Omega})$ is given explicitly. In the case of planar domain $\Omega$, using the module tensor product, the dilation for every two dimensional contractive module over $\mathcal{O}(\overline{\Omega})$ is described. The question of explicitly describing the dilations for two dimensional modules over the algebra $\mathcal{O}(\overline{\Omega})$, for any domain $\Omega \subset \mathbb{C}^m$ is also investigated.

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